Metric spaces

The definition of a topological space is very abstract and can seem unintuitive w/ out an example to keep in mind. We'll start out w/ the much easier to swallow example of a metric space before jumping into pure abstraction.

Def: A <u>metric space</u> (X,d) is a set X, together w/ a function $d : X \times X \to \mathbb{R}^{\geq 0}$, called the <u>distance</u> <u>function</u>, satisfying the conditions

1.) For $p,q \in X$, $d(p,q) = 0 \iff p = q$,

2.) $p,q \in X \Rightarrow d(p,q) = d(q,p)$,

3.) (triangle inequality) $p,q,r \in X$

$\Rightarrow d(p,r) \leq d(p,q) + d(q,r)$.

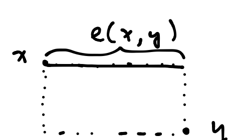
Ex: $X = \mathbb{R}^n$ w/ the usual distance:

$$d(\vec{x},\vec{y}) = \sqrt{\sum (y_i - x_i)^2}$$

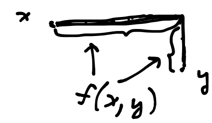
If $Y \subseteq \mathbb{R}^n$, then $(Y, d|_Y)$ is a metric space ($d|_Y$ called the <u>induced</u> <u>metric</u>)

We can also put different metrics on $\mathbb{R}^n$:

Ex: $e(\vec{x},\vec{y}) = \max(|y_i - x_i|)$ (square metric)



or $f(\vec{x},\vec{y}) = \sum |y_i - x_i|$ (diamond metric)



On HW: Show $(\mathbb{R}^n, e)$, $(\mathbb{R}^n, f)$ are metric spaces.

Limits

Let (X,d) be a metric space.

Def: An infinite sequence $p_1, p_2, \ldots \in X$ has limit $p \in X$ if

$$\forall \ \varepsilon > 0, \ \exists \ N \ \text{s.t.} \ \forall \ n > N, \ d(p_n, p) < \varepsilon.$$

On hw: Any sequence of points in a metric space has at most one limit point.

Def: $p_1, p_2, \ldots \in X$ is a Cauchy sequence if

$$\forall \ \varepsilon > 0 \ \exists \ N \ \text{s.t.} \ \forall \ m,n > N \ \ d(p_m, p_n) < \varepsilon$$

On HW: If a sequence has a limit, it's Cauchy, but not the other way around.

A metric space is complete if every Cauchy sequence has a limit.

Open sets

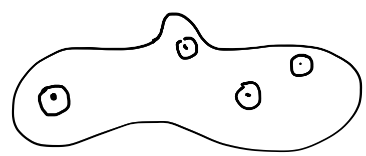
Def: (X,d) a metric space, $p \in X$, $r > 0$.

the <u>open ball</u> of radius r around p is

$$B_r(p) = \{ q \in X \mid d(p,q) < r \}$$

Def: 1) $U \subseteq X$ is <u>open</u> if $\forall\ p \in X$, $\exists\ r > 0$ s.t.

$$B_r(p) \subseteq U.$$



2.) $Z \subseteq X$ is closed if $X \setminus Z \subseteq X$ is open.

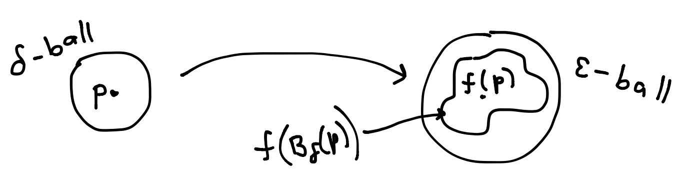
Claim (on HW): The union of arbitrarily many open sets is open, and the intersection of finitely many open sets is open.

Now we can define continuous functions w/ out explicitly using the metric (i.e. no ε, δ).

First, Recall the standard def of continuity:

Def: Let $(X, d), (Y, e)$ be metric spaces. Then $f: X \to Y$ is <u>continuous</u> if $\forall\ p \in X$, $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$$d(p,x) < \delta \implies e(f(p), f(x)) < \varepsilon$$



Theorem: $f: X \to Y$ is continuous iff $\forall$ open $U \subset Y$, $f^{-1}(U)$ is open in X.

Pf: First assume $f: X \to Y$ is continuous.

Let $U \subset Y$ be open. Let $p \in f^{-1}(U)$.

We can find a ball $B_\varepsilon(f(p)) \subseteq U$ for some $\varepsilon > 0$.

so $f^{-1}(U)$ contains a δ-ball around p. $\Rightarrow f^{-1}(U)$ is open.

For the other direction: Assume f is not continuous at p. There's some ε s.t. $f^{-1}(B_\varepsilon(f(P)))$ doesn't contain an open ball around p, so $f^{-1}(B_\varepsilon(f(P)))$ is not open.